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STOCHASTIC SINGLE PERIOD INVENTORY  
DECISIONS: BASED ON FULL QUADRATIC  
COST FUNCTIONS

by

Susan B. Padgett

June, 1994

Thesis Advisor:

Glenn F. Lindsay

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Stochastic Single Period Inventory Decisions Based On  
Quadratic Cost Functions

by

Susan B. Padgett  
Lieutenant, United States Navy  
B.S., Temple University, 1984

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JUNE 1994

Author:

*Susan B. Padgett*

Susan B. Padgett

Approved by:

*Glenn F. Lindsay*

Glenn F. Lindsay, Thesis Advisor

*Barbara Marsh-Jones*

Barbara Marsh-Jones, Second Reader

*P. Purdue*

Peter Purdue, Chairman

Department of Operations Research

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## ABSTRACT

This study addresses a general class of decision situations whose solutions are directly applicable to military inventory acquisitions and or disposals. Although optimal solutions are well known when subsequent costs are linear to the amount of surplus or shortage, the perhaps more realistic case of non-linear costs has not been extensively studied. The results of this study suggest the optimal solutions, i.e., acquisition quantity or supply, for both conditions of risk and uncertainty about demand when the associated cost function is non-linear and quadratic. For conditions of risk, optimal solutions are found which will yield minimum expected costs for the two-piece cost function where surplus and shortage costs are quadratic. This is done for both discrete and continuous demand variables. When future need for the item is unknown and only the maximum value can be estimated, optimal solutions are obtained for goals of minimaxing cost, minimaxing regret, and the Laplace criteria using a uniform probability distribution. It is shown that these different approaches to determining acquisition quantities under conditions of uncertainty lead, for this general class of decision problems, to the same optimal result. Hopefully, this information will aid in the decision process while making affordability assessments of new acquisitions.

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## **EXECUTIVE SUMMARY**

This study addresses a general class of decision situations whose solutions are directly applicable to inventory acquisitions and or disposals. Although optimal solutions are well known when subsequent costs are linear to the amount of surplus or shortage, the perhaps more realistic case of non-linear costs has not been extensively studied.

Even the most casual observer of the defense marketplace would agree that long-term growth in the costs of military hardware has been substantial. Agencies, such as the Center for Naval Analysis, are often involved with making affordability assessments of new acquisitions under the Department of Defense. In determining the affordability of a new acquisition, the entire life-cycle costs need to be evaluated. When acquiring a major piece of military hardware, costs associated with items such as spares or replacement parts can consume a significant portion of the total costs. The timing of when resources are expended plays a crucial role in determining the affordability of a new program. Spares or replacement parts not purchased at acquisition time but needed (and purchased) later, tend to cost much, much more. Furthermore, when the particular piece of hardware is at the forefront of new technology the reliability of it or its components may be unknown. Estimates of the probability distribution of demand for the number of replacement

parts over the life of the hardware will have to be considered along with the associated cost functions.

The results of this study suggest the optimal solutions, i.e., acquisition quantity or supply, for both conditions of risk and uncertainty about demand when the associated cost function is non-linear and quadratic. This is done for two-piece cost functions where each part (surplus and shortage) recognizes both linear and quadratic terms. The optimal solutions found for conditions of risk will yield the minimum expected costs. For the continuous case, it is derived by differentiating the expected cost equation with respect to the decision variable, i.e., the acquisition quantity, and setting the result equal to zero. For the discrete case, it is found that sufficiency conditions for the local minimum of a function (finite difference equations) are elemental in deriving the optimal acquisition quantity. When future need for the item is unknown and only the maximum value can be estimated, optimal solutions are obtained for goals of minimaxing cost, minimaxing regret, and the Laplace criteria using a uniform probability distribution.

This analysis of a quadratic cost equation has not been studied extensively in the past. Hopefully, this information will aid in the decision process while making affordability assessments of new acquisitions.

## I. INTRODUCTION

Stochastic single-period inventory models, more commonly known as "Newspaper Boy" problems, first made an appearance in 1950, where they were described in Morse and Kimball's early Operations Research text [Ref.1]. Based on certain assumptions about demand, the problem involves making inventory acquisition decisions with the objective to minimize expected cost. These models are directly applicable to determining the optimal quantity of spares to acquire with a new system [Ref. 2].

When acquiring a major piece of military hardware, costs associated with spares or replacement parts are a significant portion of the total costs. Replacement items not purchased at acquisition time but needed later, tend to cost much, much more. Furthermore, when the particular piece of hardware is at the forefront of new technology the reliability of it or its components may be unknown. Estimates of the probability distribution of demand for the number of replacement parts over the life of the hardware will have to be considered along with the associated cost functions. Estimating the demand uncertainty was researched in a study by Pranom Srinopakoon [Ref. 3]. His study suggest some procedures which can be used to estimate the demand distribution even when data on unsatisfied demands are not available. In another study, a decision procedure for the transition from uncertainty

to risk was proposed by Kadir Sagdic [Ref.4] who explores the use of order-statistic-based quantile estimators as a decision procedure while data accumulates in the early decision periods.

In existing applications, cost functions have been assumed to be linearly proportional to quantity. Although optimal solutions are well known when subsequent costs are linear to the amount of surplus or shortage the perhaps more realistic case of non-linear costs has not been extensively studied. Little research has been done when associated costs are non-linear. However, even the most casual observer of the defense marketplace would agree that long-term growth in the cost of military hardware has been, and maybe expected to be, substantial. The purpose of the study reported in this thesis is to seek and examine solutions where costs behave in a non-linear manner, i.e., quadratically, as functions of shortage or surplus.

Chapter II briefly reviews the standard model where costs are linearly proportional to surplus or shortage. Chapter III is devoted to the purpose of this thesis, i.e., to the quadratic extension of the newsboy problem. Hence, optimal acquisition decisions will be sought for both conditions of risk and uncertainty about demand. Three principles of choice will be applied to decisions under uncertainty: the Laplace principle, minimax cost solutions, and minimax regret solutions. Conclusions and recommendations are given in the final chapter.

## **II. SINGLE-PERIOD INVENTORY MODEL (LINEAR COST FUNCTIONS)**

In this section we shall briefly describe the well-known newsboy problem, illustrating its optimal solutions under both risk and uncertainty with respect to demand. This generalized problem appears frequently in a variety of scenarios. The model may represent the inventory of an item that (1) becomes obsolete quickly, such as the daily newspaper; (2) spoils quickly, such as vegetables or loaves of bread; (3) is stocked only once, such as spare parts for a single production run of a new piece of military hardware; or (4) has a future that is uncertain beyond a single period, such as designer clothing fads [Ref. 5].

In general the inventory model has been structured such that orders are placed once for purchase of inventory to cover a single period. Stock shortages may not be refilled and stock surplus may not be transferred for use in the next period. Costs incurred by shortage or by surplus are proportional to the difference between the quantity on hand and the subsequent demand during the period. If exact demand were known it would be a trivial matter to minimize cost: one would simply acquire that exact amount. Unfortunately, since demand is typically unknown at the time of acquisition, and could be characterized as a random variable, then minimizing expected cost becomes a more complicated task.

However, for an appropriate decision rule, an optimal inventory policy can be obtained even if limited information about the probability distribution of demand is available. It is clear that ordering some amount more than the possible minimum demand would be desirable, but certainly the order should be no more than the maximum demand.

The single-period cost equation may be structured from the following components.

$Q$  = Quantity (supply) of inventory initially to be on hand for the period. This is the decision variable for the problem.

$D$  = Demand for the inventory during the period. Demand is unknown at the time supply  $Q$  is selected, and may usefully be considered as a discrete or continuous random variable.

$C_s$  = Cost of surplus for each unit of inventory unsold, i.e., when supply  $Q$  exceeds demand  $D$ .

$C_o$  = Cost of outage for each unit of unsatisfied demand, i.e., when demand  $D$  exceeds supply  $Q$ .

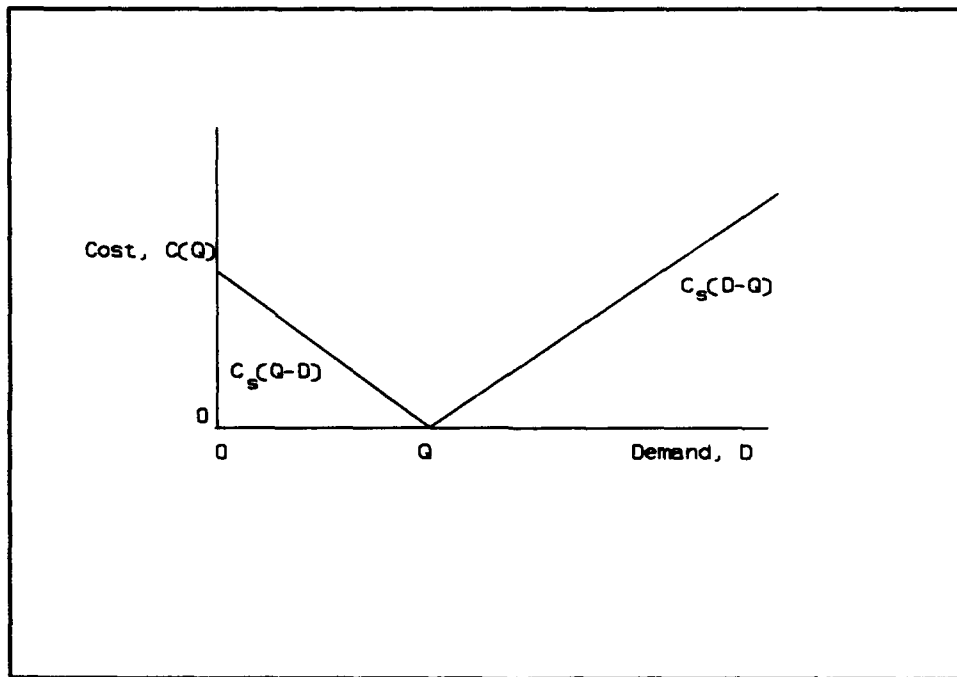
$p(d)$  = Probability distribution of demand, if demand is discrete.

$f(d)$  = Probability density function of demand, if demand is continuous.

The cost function, depending on whether there is a surplus or a shortage is

$$Cost = C(Q) = \begin{cases} C_s(Q-D), & 0 \leq D \leq Q, \\ C_o(D-Q), & Q < D. \end{cases} \quad (1)$$

For simplicity in familiarization with the basic model we'll assume that both  $Q$  and  $D$  are continuous variables. The discrete version will be explained in the next chapter, dealing with the non-linear cost function. The linear cost function appears graphically as in Figure 1.



**Figure 1. The Newsboy Cost Function**

Given a certain density function  $f(d)$  for the continuous random variable of demand, and seeking a value for  $Q^*$ , the decision variable that the decision maker is trying to find over the range between zero and  $D_{\max}$ , the expected cost is



$$E[C(Q)] = \int_0^{\infty} (Cost) f(D) dD.$$

This, upon including the cost function (1) becomes,

$$E[C(Q)] = \int_0^Q C_s(Q-D) f(D) dD + \int_Q^{\infty} C_o(D-Q) f(D) dD. \quad (2)$$

We seek the value  $Q^*$ , which minimizes this expected cost function (2). The optimal value of  $Q$  may be found by differentiating the expected cost with respect to  $Q$ , equating the derivative to zero, and solving for the optimum value  $Q^*$ . Since  $Q$  appears in the limits we may use Leibnitz rule [Ref. 8] for integration, or

$$\frac{dE[C(Q)]}{dQ} = \int_0^Q C_s f(D) dD + \int_Q^{\infty} -C_o f(D) dD = 0.$$

Applying the cumulative distribution function

$$F(x) = \int_{-\infty}^x f(t) dt$$

to obtain

$$C_s F(Q^*) - C_o [1 - F(Q^*)] = 0,$$

we arrive at the well-known result [Ref. 5]

$$F(Q^*) = \frac{C_o}{C_s + C_o} . \quad (3)$$

To verify that the expected value of cost is minimized at  $Q^*$  we should examine the second derivative

$$\left. \frac{d^2 E[C(Q)]}{dQ^2} \right|_{Q=Q^*} = (C_s + C_o) f(Q^*) .$$

Since the quantity  $(C_s + C_o)f(Q^*)$  is positive the value indeed is minimized.

To summarize, the above result says that in order to minimize the expected value of cost, the decision maker should order the quantity  $Q^*$  which is equal to the  $[C_o/(C_s + C_o)]$ th quantile of the demand distribution, as shown in Figure 2. [Ref. 6]

Furthermore, as a decision tool, it is not necessary to know the exact numerical values of  $C_s$  and  $C_o$ , since the ratio of the two may be sufficient. For example, suppose that as a best cost estimate a unit of surplus will cost twice as much as a unit of shortage, i.e.,  $C_s = 2C_o$ , which equates to determining the value for the .33 quantile estimate.

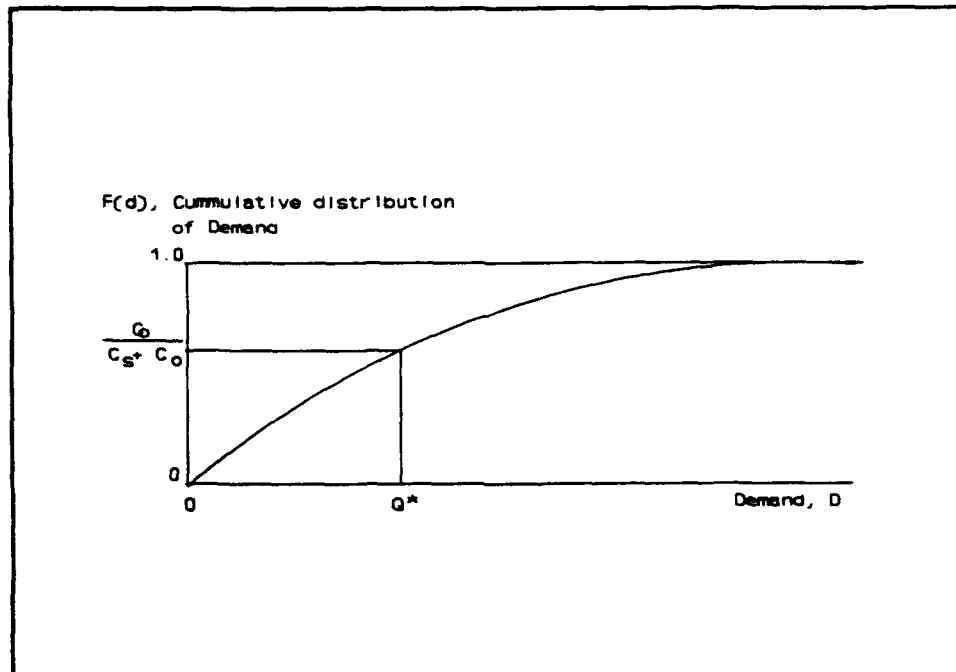
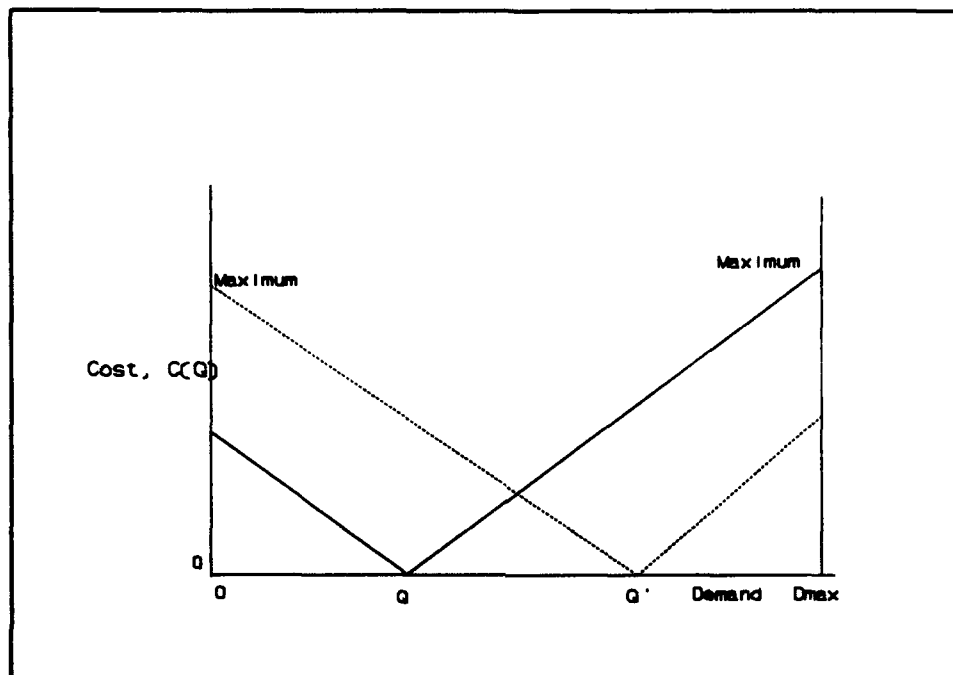


Figure 2. Minimum Expected Cost Solution

When the distribution of demand is unknown the decision becomes one under *uncertainty*. Three well-known approaches to decision making under uncertainty are the Laplace solution under the assumption of a uniform distribution of demand, minimax cost solutions and minimax regret solutions. For the newsboy problem, these all lead to the same rule to find the optimal solution. We shall discuss only the minimax cost approach. [Ref. 8]

This approach is to choose  $Q$  so that the worst possible cost will be minimized, that is, to minimize maximum cost. As is evident from the solid line in Figure 3 the maximum cost occurs at  $D = D_{\max}$ . By increasing  $Q$  to say  $Q'$  we can reduce the cost at  $D = D_{\max}$ , as shown by the dashed line in Figure 3. Note,

however, that the maximum cost now occurs when demand is at  $D=0$  and so the process continues.



**Figure 3. Illustration Of Extreme Point Maximums Of The Newsboy Cost Function With Bounded Demand.**

Clearly, we would want to choose  $Q$  so that the cost at  $D = D_{\max}$  is equal to the cost at  $D = 0$ . Equating these, we obtain

$$C_s Q^* = C_o (D_{\max} - Q^*)$$

or

$$Q^* = \left( \frac{C_o}{C_s + C_o} \right) D_{\max} \quad (4)$$

as the order quantity  $Q^*$  which minimaxes cost.

In this chapter we have familiarized ourselves with the minimum expected cost solutions under both risk and uncertainty conditions of the basic newsboy problem. We will now proceed to the more extensive case where associated costs are non-linearly proportional to the amount of surplus or shortage on hand. This will be done for both continuous and discrete quantities of supply and demand.

### III. SINGLE-PERIOD INVENTORY MODEL (QUADRATIC COST FUNCTIONS)

This chapter is devoted to the quadratic extension of the standard single-period model, i.e., the newsboy problem. Costs are no longer assumed to be linearly proportional to the difference between the quantity on hand at the beginning of the period and the subsequent amount demanded throughout the period. Instead, costs are assumed to increase quadratically in relation to a surplus or shortage. With the exception of this new cost relation the other characteristics of the standard model remain unchanged; i.e., orders are placed once for the purchase of inventory to cover a single period, stock shortages may not be refilled and stock surpluses may not be transferred for use in the next period. The cost function may be written as two second degree polynomials in the form of

$$Cost = C(Q) = \begin{cases} C_{s_1} (Q-D)^2 + C_{s_2} (Q-D), & 0 \leq D \leq Q, \\ C_{o_1} (Q-D)^2 + C_{o_2} (Q-D), & Q < D, \end{cases} \quad (5)$$

for the continuous case, and

$$Cost = C(Q) = \begin{cases} C_{s_1} (Q-D)^2 + C_{s_2} (Q-D), & D=0, 1, \dots, Q \\ C_{o_1} (D-Q)^2 + C_{o_2} (D-Q), & D=Q+1, \dots \end{cases} \quad (6)$$

for the discrete case.

The components for the quadratic cost equation are as follows:

$Q$  = Quantity (supply) of inventory initially on hand for the period.  
This is the decision variable for the problem.

$D$  = Demand for the inventory during the period. Demand is unknown at the time supply  $Q$  is selected, and may be usefully considered as a discrete or continuous random variable.

$C_{s1}$  = Cost of surplus for each squared unit of inventory unsold, i.e., when supply  $Q$  exceeds demand  $D$ .

$C_{s2}$  = Cost of surplus for each single unit of inventory unsold.

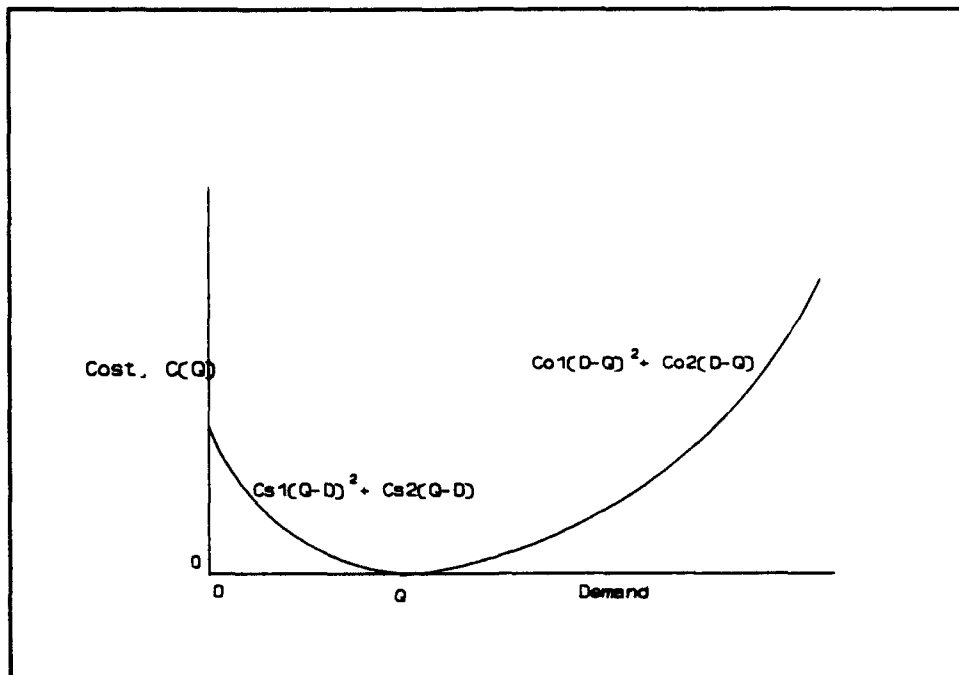
$C_{o1}$  = Cost of outage for each squared unit of unsatisfied demand, i.e., when demand  $D$  exceeds supply  $Q$ .

$C_{o2}$  = Cost of outage for each single unit of unsatisfied demand.

The cost coefficients may assume both positive and negative values. However our work will address cases where the four cost coefficients are non-negative.<sup>1</sup> Under these conditions of a quadratic cost equation the penalty of having a surplus or shortage is much greater than it is when the cost equation is linear. Total cost increases rapidly when the quantity ordered does not coincide with the subsequent demand. Notice that the cost function is a strictly convex function and may appear as in Figure 4.

---

<sup>1</sup>Cases where  $C_{s2}=C_{o2}=0$  have been investigated [Ref. 10, 11].



**Figure 4. Illustration Of The Quadratic Cost Function**

## **A. SOLUTIONS UNDER RISK**

Our ability to find good solutions for the optimal order quantity depends upon the amount of information we have about the future; in particular, about the magnitude of demand  $D$ . In this section we will assume that the probability distribution of demand is known. Our goal is to find the value of the decision variable which minimizes expected cost.

### **1. Minimizing Expected Cost, Continuous Case**

Given that demand  $D$  is a continuous random variable with a known density function  $f(d)$ , and that  $Q$  (the decision variable for the optimal order



quantity) is also a continuous variable; the expected cost equation is

$$E[ C(Q) ] = \int_{D=0}^{\infty} (Cost) f(D) dD.$$

Which, upon including the cost function (5) becomes

$$\begin{aligned} E[ C(Q) ] = & \int_{D=0}^Q C_{s_1} (Q-D)^2 f(D) dD \\ & + \int_{D=0}^Q C_{s_2} (Q-D) f(D) dD \\ & + \int_{D=Q}^{\infty} C_{o_1} (D-Q)^2 f(D) dD \\ & + \int_{D=Q}^{\infty} C_{o_2} (D-Q) f(D) dD. \end{aligned} \tag{7}$$

As with the standard single-period model, the optimal value of  $Q$  may be found by differentiating the expected cost equation with respect to  $Q$  and then equating the derivative to zero. Since  $Q$  appears in the limits of integration, we may use Liebnitz' rule for differentiation of integrals [Ref 7]. The result is

$$\begin{aligned}
\frac{dE[C(Q)]}{dQ} &= \int_{D=0}^Q 2C_{S_1} (Q-D) f(D) dD + C_{S_1} (Q-Q)^2 f(Q) \\
&+ \int_{D=0}^Q C_{S_2} f(D) dD + C_{S_2} (Q-Q) f(Q) \\
&+ \int_{D=Q}^{\infty} (-2) C_{\alpha_1} (D-Q) f(D) dD + C_{\alpha_1} (Q-Q)^2 f(D) \\
&+ \int_{D=Q}^{\infty} -C_{\alpha_2} f(D) dD + C_{\alpha_2} (Q-Q) f(D) = 0.
\end{aligned}$$

Which further simplifies to

$$\begin{aligned}
\frac{E[C(Q)]}{dQ} &= 2C_{S_1} Q F(Q) - 2C_{S_1} \int_{D=0}^Q D f(D) dD \\
&+ C_{S_2} F(Q) \\
&+ 2C_{\alpha_1} Q [1-F(Q)] - 2C_{\alpha_1} \int_{D=Q}^{\infty} D f(D) dD \\
&- C_{\alpha_2} [1-F(Q)] = 0.
\end{aligned}$$

By adding and subtracting the quantity

$$2C_{\alpha_1} \int_{D=0}^Q D f(D) dD,$$

the final result is

$$\begin{aligned}
& F(Q) [2Q(C_{s_1} - C_{o_1}) + C_{s_2} + C_{o_2}] + 2C_{o_1}(Q - E[D]) \\
& - C_{o_2} - 2(C_{s_1} - C_{o_1}) \int_{D=0}^Q D f(D) dD = 0.
\end{aligned} \tag{8}$$

The expected cost is minimized for the value of  $Q = Q^*$  which satisfies this condition.

Depending upon the distribution, solving (8) for  $Q^*$  may be a tedious task accomplished best by a series of iterations. On the other hand, the distribution may allow us to be able to solve for  $Q^*$  by simple manipulation. As an example, suppose that an elemental replacement part is available for a new piece of military hardware where the mark-up for this part is relatively high and the penalty for being out of it is unforgiven by the consumer. Let's suppose that each unit of shortage (in thousands of dollars) costs eight for a single unit with an additional cost of two for missing the subsequent demand, i.e.,  $C_{o_1} = 2.00$  and  $C_{o_2} = 8.00$ . Each unit of surplus (in thousands of dollars) costs one and its additional cost is ten percent for missing the subsequent demand, i.e.,  $C_{s_1} = 0.10$  and  $C_{s_2} = 1.00$ .

Thus the cost equation is

$$\text{Cost} = C(Q) = \begin{cases} 0.10 (Q-D)^2 + 1.00 (Q-D), & D=0,1,\dots,Q \\ 2.00 (D-Q)^2 + 8.00 (D-Q), & D=Q+1,\dots \end{cases}$$

We'll assume, that the item has a demand that is exponentially distributed with a

mean of 200, i.e.,  $E[D] = 200$ . By substituting the values for the coefficients and applying the following two characteristics of the exponential distribution [Ref. 11],

$$F(Q) = 1 - e^{-\frac{y}{200}}$$

and

$$\int_{y=0}^Q y \frac{1}{200} e^{-\frac{y}{200}} dy = 200 - Qe^{-\frac{Q}{200}} - 200e^{-\frac{Q}{200}},$$

to Equation (8) the result becomes

$$\begin{aligned} & [1 - e^{-\frac{Q}{200}}][2Q(-1.9) + 9] + 4(Q - 200) \\ & - 8 - 2(-1.9)[-Qe^{-\frac{Q}{200}} - 200e^{-\frac{Q}{200}} + 200] = 0. \end{aligned}$$

When simplified, this expression reduces to the following equality:

$$0.20Q - 769 e^{-\frac{Q}{200}} = 39 .$$

As we can see from Table 1, the value for  $Q$  which satisfies this equality lies between 504 and 505 units. Obviously, under the above constraints, it is less costly to have a surplus than it is to have a shortage. However, it is

interesting to note that here the optimal order quantity is more than two and a half times the expected demand quantity.

**Table I. Optimal Order Quantity For An Exponential Demand Distribution**

---

Q	$.2Q - 769e^{-Q/200}$
200	-242.89
400	-24.07
450	8.94
500	36.87
504	38.92
505	39.43
510	41.95
550	60.83

---

In comparison, if there are no costs associated with the quadratic terms, i.e., the model is a simple "Newspaper Boy" problem which has a linear cost function, then the optimal order quantity would be approximately 440 units. This value is also much greater than the value for the expected demand; however, it is somewhat less than the optimal value for the quadratic cost function. The quadratic solution requires a larger supply quantity because the costs associated with a shortage are

more substantial than the costs associated with a surplus.

## 2. Minimizing Expected Cost, Discrete Case

In most situations inventory exchanges are measured in discrete units where quantities are too small to permit approximation by continuous variables. In this case, both  $Q$  and  $D$  are treated as discrete variables, where  $p(D)$  is the probability distribution of demand. The expected cost for ordering  $Q$  items  $E[C(Q)]$ , is

$$E[C(Q)] = \sum_{D=0}^{\infty} (\text{Cost}) p(d).$$

This, upon including the discrete cost Equation (6), becomes

$$E[C(Q)] = \sum_{D=0}^Q [C_{s_1}(Q-D)^2 p(D) + C_{s_2}(Q-D) p(D)] + \sum_{D=Q+1}^{\infty} [C_{o_1}(D-Q)^2 p(D) + C_{o_2}(D-Q) p(D)]. \quad (9)$$

In terms of the expected cost function, sufficient conditions for a local minimum of a function for  $Q = Q^*$  and defined for integer values of its argument are

$$E[C(Q^*+1)] - E[C(Q^*)] \geq 0 \quad (10)$$

and

$$E[C(Q^* - 1)] - E[C(Q^*)] > 0. \quad (11)$$

The forward difference, Equation (10), and the backward difference, Equation (11), will be used to construct the decision rule to minimize expected cost.

Beginning with the forward difference, the expected cost, using Equation (9), at  $Q+1$  is

$$\begin{aligned} E[C(Q+1)] = & \sum_{D=0}^{Q+1} C_{s_1} (Q+1-D)^2 p(D) + \\ & \sum_{D=0}^{Q+1} C_{s_2} (Q+1-D) p(D) + \\ & \sum_{D=Q+2}^{\infty} C_{o_1} (D-Q-1)^2 p(D) + \\ & \sum_{D=Q+2}^{\infty} C_{o_2} (D-Q-1) p(D). \end{aligned}$$

When expanded this becomes,

$$\begin{aligned} E[C(Q+1)] = & \sum_{D=0}^{Q+1} C_{s_1} (Q-D)^2 p(D) + \sum_{D=0}^{Q+1} 2C_{s_1} (Q-D) p(D) + \sum_{D=0}^{Q+1} C_{s_1} p(D) \\ & + C_{s_2} (Q+1-Q-1) p(D) + \sum_{D=0}^Q C_{s_2} (Q+1-D) p(D) + \\ & \sum_{D=Q+2}^{\infty} C_{o_1} (D-Q)^2 p(D) - \sum_{D=Q+2}^{\infty} 2C_{o_1} (D-Q) p(D) + \sum_{D=Q+2}^{\infty} C_{o_1} p(D) \\ & + \sum_{D=Q+2}^{\infty} C_{o_2} (D-Q) p(D) - \sum_{D=Q+2}^{\infty} C_{o_2} p(D). \end{aligned}$$

If the expected cost at  $Q$  is truly a minimum, then the difference between the expected cost at  $Q+1$  and the expected cost at  $Q$  must be non-negative. Therefore, upon applying Equation (10), the above result (after some effort) becomes

$$2\left[\sum_{D=0}^Q C_{s_1}(Q-D) p(D) - \sum_{D=Q+1}^{\infty} C_{o_1}(D-Q) p(D)\right] \\ + (C_{s_1} + C_{s_2}) \sum_{D=0}^Q p(D) + (C_{o_1} - C_{o_2}) \sum_{D=Q+1}^{\infty} p(D) \geq 0 .$$

By using the well known identity

$$P(Q) = \sum_{D=0}^Q p(D) = 1 - \sum_{D=Q+1}^{\infty} p(D),$$

the equation simplifies to

$$2\left[\sum_{D=0}^Q C_{s_1}(Q-D) p(D) - \sum_{D=Q+1}^{\infty} C_{o_1}(D-Q) p(D)\right] \quad (12) \\ \geq (-C_{s_1} - C_{s_2} + C_{o_1} - C_{o_2})P(Q) + (C_{o_2} - C_{o_1}).$$

Equation (12) satisfies the first half of the sufficient conditions for the construction of the decision rule.

Next, we'll apply the backward difference. Using Equation (9) again, the expected cost at  $Q-1$  is



$$\begin{aligned}
E[C(Q-1)] = & \sum_{D=0}^{Q-1} C_{s_1}(Q-1-D)^2 p(D) + \\
& \sum_{D=0}^{Q-1} C_{s_2}(Q-1-D) p(D) + \\
& \sum_{D=Q}^{\infty} C_{o_1}(D-Q+1)^2 p(D) + \\
& \sum_{D=Q}^{\infty} C_{o_2}(D-Q+1) p(D) .
\end{aligned}$$

When expanded and simplified this becomes,

$$\begin{aligned}
E[C(Q-1)] = & \sum_{D=0}^Q C_{s_1}(Q-D)^2 p(D) - 2 \sum_{D=0}^Q C_{s_1}(Q-D)p(D) + \sum_{D=0}^{Q-1} C_{s_1}p(D) \\
& + \sum_{D=0}^Q C_{s_2}(Q-D) p(D) - \sum_{D=0}^{Q-1} C_{s_2} p(D) + \\
& \sum_{D=Q}^{\infty} C_{o_1}(D-Q)^2 p(D) + 2 \sum_{D=Q}^{\infty} C_{o_1}(D-Q)p(D) + \sum_{D=Q}^{\infty} C_{o_1} p(D) \\
& + \sum_{D=Q}^{\infty} C_{o_2}(D-Q)p(D) + \sum_{D=Q}^{\infty} C_{o_2}p(D).
\end{aligned}$$

As before, if the expected cost at  $Q$  is truly a minimum, then the difference between the expected cost at  $Q-1$  and the expected cost at  $Q$  must be positive.

Applying Equation (11), the above result becomes

$$\begin{aligned}
& (C_{s_1} - C_{s_2}) \sum_{D=0}^{Q-1} P(D) + (C_{o_1} + C_{o_2}) \sum_{D=Q}^{\infty} P(D) \\
& > 2 \left[ \sum_{D=0}^Q C_{s_1} (Q-D) P(D) - \sum_{D=Q+1}^{\infty} C_{o_1} (D-Q) P(D) \right].
\end{aligned}$$

By using another well known identity

$$P(Q-1) = \sum_{D=0}^{Q-1} P(D) = 1 - \sum_{D=Q}^{\infty} P(D),$$

the equation further simplifies to

$$\begin{aligned}
& (C_{s_1} - C_{s_2} - C_{o_1} - C_{o_2}) P(Q-1) + (C_{o_1} + C_{o_2}) \\
& > 2 \left[ \sum_{D=0}^Q C_{s_1} (Q-D) P(D) - \sum_{D=Q+1}^{\infty} C_{o_1} (D-Q) P(D) \right].
\end{aligned} \tag{13}$$

Equation (13) satisfies the second half of the sufficient conditions for the construction of the decision rule. Finally, by combining Equations (12) and (13), the first order difference conditions, we arrive at the decision rule

$$\begin{aligned}
& (C_{s_1} - C_{s_2} - C_{o_1} - C_{o_2}) P(Q-1) + (C_{o_1} + C_{o_2}) > \\
& 2 \left[ \sum_{D=0}^Q C_{s_1} (Q-D) P(D) - \sum_{D=Q+1}^{\infty} C_{o_1} (D-Q) P(D) \right] \geq \\
& (-C_{s_1} - C_{s_2} + C_{o_1} - C_{o_2}) P(Q) + (C_{o_2} - C_{o_1}) .
\end{aligned} \tag{14}$$

The optimal value of  $Q = Q^*$  which satisfies Equation (14) will yield the minimum expected cost solution to Equation (9).

Finding the value of  $Q$  which satisfies this double inequality may be a rather intractable task. In some cases, rather than use (14) it may be easier to handle the discrete case by direct enumeration of expected costs for each possible value of order quantity  $Q$ . For example, suppose that  $C_{s1} = 2$ ,  $C_{s2} = 4$ ,  $C_{o1} = 3$ ,  $C_{o2} = 6$ , and that the probability distribution of demand and cumulative probability function of demand are, respectively,

$$p(D) = \begin{cases} 0.1 & D=0, \\ 0.2 & D=1, \\ 0.4 & D=2, \\ 0.2 & D=3, \\ 0.1 & D=4, \end{cases} \quad \text{and} \quad P(D) = \begin{cases} 0.1 & D=0, \\ 0.3 & D=1, \\ 0.7 & D=2, \\ 0.9 & D=3, \\ 1.0 & D=4. \end{cases}$$

Values of the cost function associated with the various values of supply  $Q$ , and demand  $D$ , are as shown in Table II. For example, if supply  $Q$ , at the beginning of the period, is equal to 3 and demand  $D$ , by the end of the period, was only 1, the cost would be

$$\text{Cost}(Q,D) = C(3,1) = 2(3-1)^2 + 4(3-1) = 16.$$

**Table II.** Cost Matrix For A Discrete Quadratic Cost Function.

		DEMAND D				
S U P P L Y  Q		0	1	2	3	4
	0	0	9	24	45	72
	1	6	0	9	24	45
	2	16	6	0	9	24
	3	30	16	6	0	9
	4	48	30	16	6	0

Notice that the cost matrix is an upper and lower semi-symmetric matrix with zeros on the diagonal. The expected cost  $E[C(Q)]$  may be computed for each of the five candidate values of  $Q$  as

$$E[C(0)] = 0(.1) + 9(.2) + 24(.4) + 45(.2) + 72(.1) = 27.60$$

$$E[C(1)] = 6(.1) + 0(.2) + 9(.4) + 24(.2) + 45(.1) = 13.50$$

$$E[C(2)] = 16(.1) + 6(.2) + 0(.4) + 9(.2) + 24(.1) = 7.00$$

$$E[C(3)] = 30(.1) + 16(.2) + 6(.4) + 0(.2) + 9(.1) = 9.50$$

$$E[C(4)] = 48(.1) + 30(.2) + 16(.4) + 6(.2) + 0(.1) = 18.40.$$

We conclude that, by direct enumeration, the optimal supply value is

$Q^* = 2$ . Notice that, even though the cost for having a shortage is significantly greater than it is for having a surplus, in this case the optimal supply value is the same as the expected demand value. If Equation (8) were used to solve for the optimal value  $Q$  then the task would be more cumbersome. However, it should also lead to the same solution for the optimal supply value  $Q^*$ . For example, by substituting the values for the cost coefficients, Equation (14) becomes

$$-11 P(Q-1) + 9 > 4 \sum_{D=0}^Q (Q-D)p(D) - 6 \sum_{D=Q+1}^{\infty} (D-Q)p(D) \geq -9 P(Q) - 3.$$

As we can see from Table III, the optimal supply value  $Q$  that satisfies Equation (14) is  $Q^* = 2$ , which is the same as before.

**Table III. Optimal Supply Value  $Q^*$  For A Discrete Cost Equation.**

$Q$	$-11P(Q+1)+9$	$4 \sum (Q-D)p(D) - 6 \sum (D-Q)p(d)$	$-9P(Q) - 3$
0	9.0	$4(0.0) - 6(2.0) = -12$	-3.9
1	7.9	$4(0.1) - 6(1.1) = -6.2$	-5.7
2	5.7	$4(0.4) - 6(0.4) = -0.8$	-9.3
3	1.3	$4(1.1) - 6(0.1) = 3.8$	-11.1
4	-0.9	$4(2.0) - 6(0.0) = 8.0$	-12.0
5	-2.0	$4(3.0) - 6(0.0) = 12.0$	12.0

## B. SOLUTIONS UNDER UNCERTAINTY

Deciding on  $Q$  when the probabilities of future demand are unknown and not estimated falls into the class called decisions under uncertainty. Here, although the probabilities are unknown, it may be possible to have some information about the distribution without actually knowing the distribution. For example, we may be able to estimate one or more descriptive values relating to the demand distribution, such as mean, mode, or median. We may have an idea about the shape of the distribution as being symmetric or perhaps skewed. We may also be able to estimate the bounds for the range of demand values; e.g., demand could range from zero to say  $D_{\max}$ . All of these provide information which may be useful in determining the optimal supply value.

We shall examine the last case, where the only estimate that can be made about the nature of the unknown demand is that its range is from zero to some upper bound  $D_{\max}$ <sup>2</sup>.

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<sup>2</sup>The general case of  $D_{\min} \leq D \leq D_{\max}$  is readily handled by noting that we would always stock  $D_{\min}$ , and the decision problem concerns the amount of supply in excess of  $D_{\min}$ .

Given this as our only information, three types of rationale (principles of choice) have been proposed for decisions under uncertainty [Ref. 8]. They are

- The Laplace Solution,
- Minimax Cost Solutions, and
- Minimax Regret Solutions.

Although different in their objectives, we will show that all three lead to the same rule for computing  $Q^*$ .

### 1. The Laplace Solutions

The Laplace approach to the case of uncertainty is to assume that demand is uniformly distributed over the demand interval and then use an expected value solution. In our case, the interval is  $0 \leq D \leq D_{\max}$ . We seek to examine both cases of continuous and discrete demand.

#### *a. Continuous case*

If demand is a continuous random variable, we would assume that the distribution is rectangular with density

$$f(D) = \frac{1}{D_{\max}}, \quad 0 \leq D \leq D_{\max}.$$



Thus, the distribution function is

$$F(D) = \frac{D}{D_{\max}}.$$

Applying this and the identity,

$$\int_{D=0}^Q D f(D) dD = \int_{D=0}^Q D \frac{1}{D_{\max}} dD = \frac{Q^2}{2 D_{\max}},$$

to the expected cost solution (8)

$$\begin{aligned} & F(Q)[2Q(C_{s_1} - C_{o_1}) + C_{s_2} + C_{o_2}] + 2C_{o_1}(Q - E[D]) \\ & - C_{o_2} - 2(C_{s_1} - C_{o_1}) \int_{D=0}^Q D f(D) dD = 0, \end{aligned} \quad (8)$$

we have

$$\begin{aligned} & \frac{Q}{D_{\max}}[2Q(C_{s_1} - C_{o_1}) + C_{s_2} + C_{o_2}] + 2C_{o_1}(Q - \frac{D_{\max}}{2}) \\ & - C_{o_2} - 2(C_{s_1} - C_{o_1}) \frac{Q^2}{2 D_{\max}} = 0. \end{aligned}$$

When expanded this becomes,

$$\left( \frac{(C_{s1} - C_{o1})}{D_{\max}} \right) Q^2 + \left( \frac{(C_{s2} + C_{o2})}{D_{\max}} + 2C_{o1} \right) Q - (C_{o1}D_{\max} + C_{o2}) = 0 ;$$

which, as we notice, is a quadratic equation in Q. This leads us to<sup>3</sup>

$$Q = \frac{-\left( \frac{C_{s2} + C_{o2}}{D_{\max}} + 2C_{o1} \right) + \sqrt{\left( \frac{C_{s2} + C_{o2}}{D_{\max}} + 2C_{o1} \right)^2 + 4 \left( \frac{C_{s1} - C_{o1}}{D_{\max}} \right) (C_{o1}D_{\max} + C_{o2})}}{2 \left( \frac{C_{s1} - C_{o1}}{D_{\max}} \right)} . \quad (15)$$

As a check, we need to ensure that the rules of mathematics are not violated, i.e., the value under the radical must be non-negative and the divisor of a fraction must not be zero. By expanding the value under the radical we get

$$\begin{aligned} & \left( \frac{C_{s2} + C_{o2}}{D_{\max}} \right)^2 + \frac{4C_{o1}(C_{s2} + C_{o2})}{D_{\max}} + 4C_{o1}^2 + \\ & \frac{4C_{s1}C_{o1}D_{\max}}{D_{\max}} + \frac{4C_{s1}C_{o2}}{D_{\max}} - \frac{4C_{o1}^2D_{\max}}{D_{\max}} - \frac{4C_{o1}C_{o2}}{D_{\max}}, \end{aligned}$$

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<sup>3</sup>The general solution for quadratic equations provides two solutions, but we are only concerned with the non-negative values of Q.

which equals,

$$\begin{aligned} & \frac{C_{s2}^2}{D_{\max}^2} + \frac{2C_{s2}C_{o2}}{D_{\max}^2} + \frac{C_{o2}^2}{D_{\max}^2} + \frac{4C_{o1}C_{s2}}{D_{\max}} + \frac{4C_{o1}C_{o2}}{D_{\max}} + \\ & 4C_{o1}^2 + 4C_{s1}C_{o1} + \frac{4C_{s1}C_{o2}}{D_{\max}} - 4C_{o1}^2 - \frac{4C_{o1}C_{o2}}{D_{\max}}. \end{aligned}$$

Notice that the last two terms cancel are cancelled by other terms and the remaining terms are all positive; therefore, the sign rule under the radical has not been violated. For the other rule, if  $C_{s1}$  equals  $C_{o1}$  then the divisor of the fraction is zero. However, if this were the case we wouldn't have a quadratic equation in  $Q$ . The solution for the optimal value of  $Q = Q^*$  would simply be

$$Q^* = \frac{C_{o1}D_{\max} + C_{o2}}{\left( \frac{(C_{s2} + C_{o2})}{D_{\max}} + 2C_{o1} \right)}.$$

In comparison, this provides a simple adjustment to the Laplace solution of a linear cost function which is

$$Q^* = \left( \frac{C_{o2}}{C_{s2} + C_{o2}} \right) D_{\max},$$

and where the coefficients of the quadratic terms are equal.

In conclusion, for the Laplace criterion, Equation (15) above provides the general means of computing the optimal supply quantity  $Q = Q^*$  which will yield the minimum expected cost for a quadratic cost function when a decision under uncertainty is made.

*b. Discrete case*

If demand is a discrete random variable, the Laplace approach would assume the uniform distribution for demand

$$f(d) = \frac{1}{D_{\max} + 1}, \quad D=0,1,\dots,D_{\max}.$$

Here, the distribution function is

$$P(D) = \frac{D + 1}{D_{\max} + 1},$$

and for this case the optimal expected value rule (14)

$$\begin{aligned} & (C_{s_1} - C_{s_2} - C_{o_1} - C_{o_2})P(Q-1) + (C_{o_1} + C_{o_2}) > \\ & 2\left[\sum_{D=0}^Q C_{s_1}(Q-D) p(D) - \sum_{D=Q+1}^{\infty} C_{o_1}(D-Q) p(D)\right] \geq \quad (14) \\ & (-C_{s_1} - C_{s_2} + C_{o_1} - C_{o_2})P(Q) + (C_{o_2} - C_{o_1}), \end{aligned}$$

leads to

$$\begin{aligned}
 & (C_{s_1} - C_{s_2} - C_{o_1} - C_{o_2}) \frac{Q^*}{D_{\max}+1} + (C_{o_1} + C_{o_2}) > \\
 & 2 \left[ (C_{s_1} - C_{o_1}) Q^* \left( \frac{Q^*+1}{D_{\max}+1} \right) + C_{o_1} Q^* + (C_{o_1} - C_{s_1}) \frac{Q^*(Q^*+1)}{2(D_{\max}+1)} - C_{o_1} \left( \frac{D_{\max}}{2} \right) \right] \geq \\
 & (-C_{s_1} - C_{s_2} + C_{o_1} - C_{o_2}) \frac{Q^*+1}{D_{\max}+1} + (C_{o_2} - C_{o_1}) .
 \end{aligned}
 \tag{16}$$

We have used the identities

$$\sum_{D=Q^*+1}^{\infty} D p(D) = E[D] - \sum_{D=0}^{Q^*} D p(D)$$

and

$$\sum_{D=0}^{Q^*} D p(D) = \frac{1}{D_{\max}+1} \frac{Q^*(Q^*+1)}{2}$$

to arrive at Equation (16). Notice that the above expression (16) can be further simplified to

$$\begin{aligned}
& (C_{s_1} - C_{s_2} - C_{o_1} - C_{o_2}) \frac{Q^*}{D_{\max} + 1} + (C_{o_1} + C_{o_2}) > \\
& (C_{s_1} - C_{o_1}) Q^* \left( \frac{Q^* + 1}{D_{\max} + 1} \right) + 2 C_{o_1} Q^* - C_{o_1} D_{\max} \geq \\
& (-C_{s_1} - C_{s_2} + C_{o_1} - C_{o_2}) \frac{Q^* + 1}{D_{\max} + 1} + (C_{o_2} - C_{o_1}) ,
\end{aligned}$$

and ultimately to

$$\begin{aligned}
& 2C_{s_1}Q^* + C_{s_2} + C_{s_1} > \\
& C_{o_1}(D_{\max} - Q^*)^2 + C_{o_2}(D_{\max} - Q^*) - C_{s_1}Q^{*2} - C_{s_2}Q^* \geq \quad (17) \\
& 2 C_{o_1}(D_{\max} - Q^*) - C_{o_1} - C_{o_2}.
\end{aligned}$$

This is the decision rule which leads to the optimal value  $Q^*$  which must simultaneously satisfy two quadratic expressions in  $Q$ , i.e.,

$$(C_{s_1} - C_{o_1})Q^{*2} + (2C_{s_1} + C_{s_2} + 2C_{o_1}D_{\max} + C_{o_2})Q + (C_{s_1} + C_{s_2} - C_{o_1}D_{\max}^2 - C_{o_2}D_{\max}) \geq 0$$

and

$$(C_{s_1} - C_{o_1})Q^{*2} + (C_{s_2} + 2C_{o_1}D_{\max} + 2C_{o_1} + C_{o_2})Q - [C_{o_1}(D_{\max} + 1)^2 + C_{o_2}(D_{\max} + 1)] > 0.$$

This value for which  $Q=Q^*$  will yield the minimum expected cost for the quadratic cost function when a Laplace decision under uncertainty is made and the only

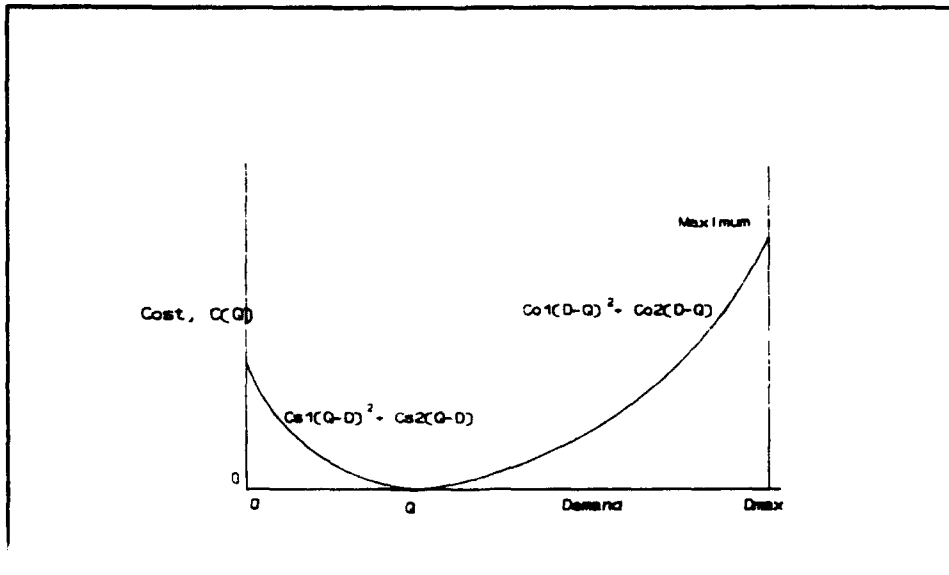
information we have about demand is that it is less than or equal to, say,  $D_{\max}$ .

## 2. Minimax Cost Solution

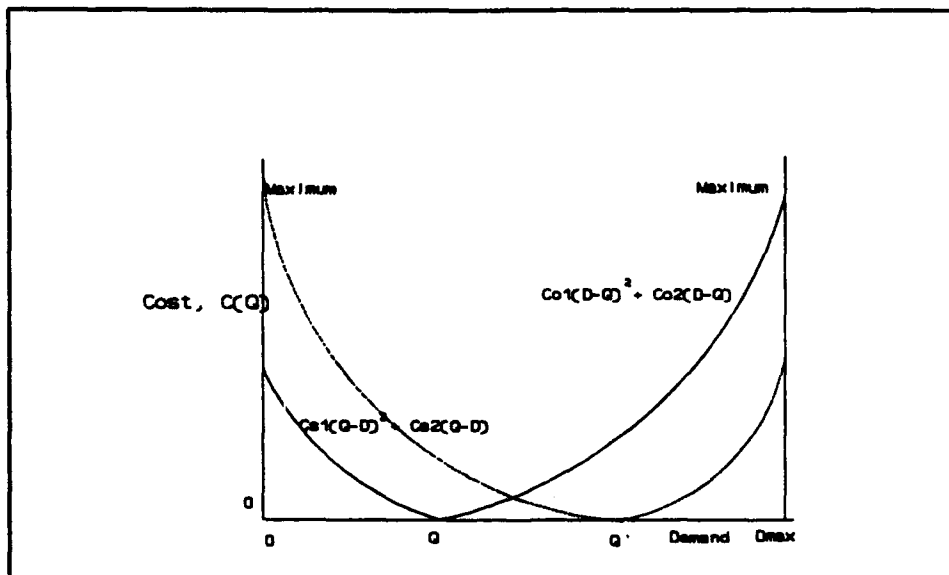
Another approach to decisions under uncertainty about future demand is to choose  $Q$  so that the worst possible future cost will be as small a possible, that is, to minimize maximum cost.

### *a. Continuous case*

For the case of continuous demand, the optimal value of  $Q$  in terms of minimaxing cost is easily found. As shown graphically in Figure 5 we can see that for the quadratic cost function at the indicated value of  $Q$ , the maximum cost will occur when demand is equal to  $D_{\max}$ . By increasing  $Q$  we can reduce the cost at  $D = D_{\max}$ . If we use a higher value, say  $Q'$ , then we will have the cost function shown by the dashed line in Figure 6. Note, however, that the maximum cost now occurs when demand is at  $D = 0$ , and to reduce this maximum possible value, we would want to reduce  $Q$  from the  $Q'$  value, and so the process continues. It should be clear that we will minimize the maximum cost when  $Q$  is chosen so that the cost at  $D = 0$  is equal to the cost at  $D = D_{\max}$ . This is the approach to the minimax cost solution.



**Figure 5. Maximum Cost In The Quadratic Cost Function With Bounded Demand**



**Figure 6. Illustration Of Extreme Point Maximums Of The Quadratic Cost Function With Bounded Demand.**



Equating the cost at the two extreme points of demand, we get

$$C_{s1}Q^{*2} + C_{s2}Q^* = C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*).$$

Notice that this is a quadratic equation in  $Q$  and by grouping the terms we have

$$(C_{s1} - C_{o1})Q^{*2} + (C_{s2} + 2D_{\max}C_{o1} + C_{o2})Q - C_{o1}D_{\max}^2 - C_{o2}D_{\max} = 0.$$

If we divide through by  $D_{\max}$  (not equal to zero) and solve for  $Q$  we have<sup>4</sup>

$$Q = \frac{-\left(\frac{C_{s2} + C_{o2}}{D_{\max}} + 2C_{o1}\right) + \sqrt{\left(\frac{C_{s2} + C_{o2}}{D_{\max}} + 2C_{o1}\right)^2 + 4\left(\frac{C_{s1} - C_{o1}}{D_{\max}}\right)(C_{o1}D_{\max} + C_{o2})}}{2\left(\frac{C_{s1} - C_{o1}}{D_{\max}}\right)}.$$

(18)

Notice that Equation (18) is identical with the Laplace solution (15) found earlier.

This is the optimal value of supply ( $Q=Q^*$ ) that yields the minimum expected cost for the quadratic cost function when a decision under uncertainty is made.

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<sup>4</sup>As before, the general solution for quadratic equations provides for two solutions, but we are only concerned with non-negative values of  $Q$ .

**b. Discrete case**

When demand is discrete, the minimax cost solution is not directly apparent. As with solutions under risk, in order for costs to be minimaxed at  $Q = Q^*$ , the following inequalities must hold:

$$\text{Maximum } C(Q^*) < \text{maximum } C(Q^* + 1), \quad (19)$$

and

$$\text{Maximum } C(Q^*) < \text{maximum } C(Q^* - 1). \quad (20)$$

As we saw in the continuous case, the cost function will have its maximum value at either  $C_{s1}(Q)^2 + C_{s2}(Q)$ , or at  $C_{o1}(D_{\max} - Q)^2 + C_{o2}(D_{\max} - Q)$ . These reflect the costs at the extreme values of  $D$ , i.e.,  $D$  equal to 0 and  $D$  equal to  $D_{\max}$ . The first condition (19) may be written as

$$\begin{aligned} & \text{MAX} [ C_{s1}(Q^*)^2 + C_{s2}(Q^*) , C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) ] < \\ & \text{MAX} [ C_{s1}(Q^* + 1)^2 + C_{s2}(Q^* + 1) , C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1) ] . \end{aligned} \quad (21)$$

Because

$$C_{s1}(Q^*)^2 + C_{s2}(Q^*) < C_{s1}(Q^* + 1)^2 + C_{s2}(Q^* + 1)$$

and

$$C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1) < C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*),$$

$C_{s1}(Q^* + 1)^2 + C_{s2}(Q^* + 1)$  is the maximum of the right-hand side of (21). (See Appendix A for further clarification.) Therefore,

$$C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) < C_{s1}(Q^* + 1)^2 + C_{s2}(Q^* + 1), \quad (22)$$

provides a useful form of the first condition for a minimax cost solution at  $Q^*$ .

Proceeding to the second condition (20), it may be written as

$$\begin{aligned} & \text{MAX} [ C_{s1}(Q^*)^2 + C_{s2}(Q^*) , C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) ] < \\ & \text{MAX} [ C_{s1}(Q^* - 1)^2 + C_{s2}(Q^* - 1) , C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1) ]. \end{aligned} \quad (23)$$

Because

$$C_{s1}(Q^* - 0)^2 + C_{s2}(Q^* - 0) > C_{s1}(Q^* - 1)^2 + C_{s2}(Q^* - 1),$$

and

$$C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1) > C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*),$$

$C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1)$  is the maximum of the right-hand side of (23).

(See Appendix A for further clarification.) Thus, we have

$$C_{s1}(Q^*-0)^2 + C_{s2}(Q^*-0) < C_{o1}(D_{\max}-Q^*+1)^2 + C_{o2}(D_{\max}-Q^*+1). \quad (24)$$

as the useful form of the second condition for a minimax cost solution at  $Q^*$ .

Together, from the first condition (21) and the second condition (24), we obtain

$$\begin{aligned} 2C_{s1}Q^* + C_{s2} + C_{s1} &> \\ C_{o1}(D_{\max}-Q^*)^2 + C_{o2}(D_{\max}-Q^*) - C_{s1}Q^{*2} - C_{s2}Q^* &> \quad (25) \\ 2C_{o1}(D_{\max}-Q^*) - C_{o1} - C_{o2}, \end{aligned}$$

as the optimal decision rule to minimax costs for decisions under uncertainty with discrete demand. Notice that the result (25) is the same as that obtained via the Laplace approach (17).

### 3. Minimax Regret Solutions

Yet another way to address decisions is through consideration of regret. The minimax regret approach to decisions under uncertainty considers the regret the decision maker will feel later if the best decision, i.e., the one which has the least cost, were not made. Thus, regret is the difference between the cost incurred by choosing the quantity  $Q$  and the lowest possible cost that could have been

obtained, had the optimal value  $Q^*$  been chosen. Therefore, for each value of demand,

$$\text{Regret}(Q) = C(Q) - \underset{Q}{\text{minimum}} [C(Q)].$$

However,

$$\underset{Q}{\text{minimum}} [C(Q)] = 0$$

for all values of demand where

$$0 \leq D \leq D_{\max},$$

and thus

$$\text{Regret}(Q) = C(Q) .$$

It follows that for the quadratic cost function, the solutions which minimax regret are the same as those which minimax cost, viz., (18) for continuous demand and (25) for discrete demand.

In summary, although different in their objectives, all three (principles of choice lead) lead to the same rule for computing the optimal value of supply  $Q$  which will minimize the expected cost when a decision under uncertainty is made. Thus we have obtained a useful rule for selecting the optimal acquisition quantity when future demand is not known.

#### IV. DISCUSSION, CONCLUSION AND RECOMMENDATIONS

Determination of acquisition quantities is of increasing importance for today's Navy. With a declining defense budget, considerable attention is being focused on various programs and capabilities. According to a recent report published by the Center for Naval Analysis, affordability assessments are being formalized and incorporated into the acquisition management process. While the principles and methods applicable to those assessments are still in a rudimentary stage, it is clear that affordability, which is concerned with top-line budget constraints and the magnitude and timing of competing uses of resources, is a *macro* concept. [Ref.12]

The timing of when resources are expended plays a crucial role in determining the affordability of a new program. As stated in the introduction, when acquiring a major piece of hardware (such as a tactical aircraft) one usually can buy spare parts at the same time. Replacement items not purchased at that time but needed later tend to cost much, much more, due to such factors as retooling, lost technology, and down time while the item is being acquired. Resources allocated to the acquisition of spare parts for a major piece of hardware can consume a significant portion of the item's total budget; therefore, making it unaffordable.

In a book written by Norman Augustine, *Augustine's Laws*, he finds a colorful way of characterizing the growth that has occurred in the cost of military aircraft:

In the year 2054, the entire defense budget will purchase just one aircraft. This aircraft will have to be shared by the Air Force and the Navy 3-1/2 days each per week except for leap year, when it will be made available to the Marines for the extra day. [Ref. 13:p.143]

The decision of the optimal order quantity of spares to purchase in support of a new system depends, of course, not only on distribution of demand but also on the expected increase in cost for those spares if purchased at a later date. In an ideal situation having just the right number of spares over the lifetime of the program will bear the minimum costs. However, if costs are predicted to increase non-linearly over time then acquiring the optimal supply of spares becomes even more crucial.

It is recognized that modelling non-linear costs as quadratic functions is only one approach. One might, for example, use cubic functions. A possible continuation of the work reported here is to explore optimal decisions for other non-linear cost relations. Another recommendation is to find optimal expressions for  $Q$  with results which would be more tractable than those found here. Another study continuing this work could address methods for solving for the coefficients  $C_{s1}$ ,  $C_{s2}$ , etc..

This study has sought the optimal quantity of supply for both conditions of

risk and uncertainty about demand, when the cost function is quadratic. This situation has not been studied extensively in the past. Hopefully the results of this thesis will provide useful insights into the decision process while making affordability assessments of new acquisitions.



## APPENDIX A

To be determined is the **maximum** value of a maximization argument defined by an inequality. From page 39 we are given

$$C_{s1}(Q^*)^2 + C_{s2}(Q^*) < C_{s1}(Q^*+1)^2 + C_{s2}(Q^*+1)$$

and

$$C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1) < C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) .$$

We wish to show that the maximum of the right hand side of

$$\begin{aligned} & \text{MAX} [ C_{s1}(Q^*)^2 + C_{s2}(Q^*) , C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) ] < \\ & \text{MAX} [ C_{s1}(Q^*+1)^2 + C_{s2}(Q^*+1) , C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1) ] , \end{aligned}$$

is

$$C_{s1}(Q^*+1)^2 + C_{s2}(Q^*+1) .$$

Proof by contradiction: suppose

$$C_{s1}(Q^*+1)^2 + C_{s2}(Q^*+1) < C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1) ,$$

then

$$C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1)$$

is greater than both

$$C_{s1}(Q^*)^2 + C_{s2}(Q^*) \quad \text{and} \quad C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*),$$

but this contradicts the one of the givens that

$$C_{o1}(D_{\max} - Q^* - 1)^2 + C_{o2}(D_{\max} - Q^* - 1)$$

is less than

$$C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) .$$

To be determined is the minimum value of a maximization argument defined by an inequality. From page 40 we are given

$$C_{s1}(Q^*-0)^2 + C_{s2}(Q^*-0) > C_{s1}(Q^*-1)^2 + C_{s2}(Q^*-1),$$

and

$$C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1) > C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) \text{ ,}$$

We wish to show that the minimum of the right hand side of

$$\begin{aligned} & \text{MAX} [ C_{s1}(Q^*)^2 + C_{s2}(Q^*) \text{ , } C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) ] < \\ & \text{MAX} [ C_{s1}(Q^*-1)^2 + C_{s2}(Q^*-1) \text{ , } C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1) ] \text{ ,} \end{aligned}$$

is

$$C_{s1}(Q^*-1)^2 + C_{s2}(Q^*-1) \text{ .}$$

Proof by contradiction: suppose

$$C_{o1}(D_{\max} - Q^* + 1)^2 + C_{o2}(D_{\max} - Q^* + 1) < C_{s1}(Q^*-1)^2 + C_{s2}(Q^*-1) \text{ ,}$$

then

$$C_{s1}(Q^*-1)^2 + C_{s2}(Q^*-1),$$

is greater than both

$$C_{s1}(Q^*)^2 + C_{s2}(Q^*) \quad \text{and} \quad C_{o1}(D_{\max} - Q^*)^2 + C_{o2}(D_{\max} - Q^*) ,$$

but this contradicts one of the givens that

$$C_{s1}(Q^* - 1)^2 + C_{s2}(Q^* - 1)$$

is less than

$$C_{s1}(Q^*)^2 + C_{s2}(Q^*) .$$

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